Matching gauge theory and string theory in a decoupling limit of AdS/CFT

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ABSTRACT: We identify a regime of the AdS/CFT correspondence in which we can quantitatively match $\mathcal{N} = 4$ super Yang-Mills (SYM) for small 't Hooft coupling with weakly coupled type IIB string theory on $\mathrm{AdS}_5 \times S^5$. We approach this regime by taking the same decoupling limit on both sides of the correspondence. On the gauge theory side only the states in the SU(2) sector survive, and in the planar limit the Hamiltonian is given by the $XXX_{1/2}$ Heisenberg spin chain. On the string theory side we show that the decoupling limit corresponds to a non-relativistic limit. In this limit some of the bosonic modes and all of the fermionic modes of the string become infinitely heavy and decouple. We first take the decoupling limit of the string sigma-model classically. This enables us to identify a semi-classical regime with semi-classical string states even though we are in a regime corresponding to small 't Hooft coupling. We furthermore analyze the quantum corrections that enter in taking the limit. From this we infer that gauge theory and string theory match, both in terms of the action and the spectrum, for the leading part and the first correction away from the semi-classical regime. Finally we consider the implications for the hitherto unexplained matching of the one-loop contribution to the energy of certain gauge theory and string theory states, and we explain how our results give a firm basis for the matching of the Hagedorn temperature in hep-th/0608115.

Contents

1. Introduction

The duality between gauge theory and string theory plays a major role in modern theoretical physics. In terms of the AdS/CFT correspondence [1–3], it is responsible for progress in understanding the non-perturbative behavior of both gauge theory and string theory. It has also led to insights concerning phenomenologically viable gauge theories (see for example the review [4]).

However, it is difficult to test the AdS/CFT correspondence directly, since the gauge theory and string theory sides usually are not applicable in the same regime. Indeed, the conventional wisdom is that one needs the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ to be large, and to be in the planar limit, in order to see strings in gauge theory, while perturbative gauge theory calculations only are valid for $\lambda \ll 1$. In [5–7] a proposal was put forward for a particular regime of AdS/CFT in which both gauge theory and string theory are reliable, and hence can be subject to a detailed match. The regime is

$$E - J \ll \lambda \ll 1, \quad J \gg 1 \tag{1.1}$$

On the gauge theory side, we are considering SU(N) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, and E is the energy of a state measured in units of the three-sphere radius, while $J = J_1 + J_2$ is the sum of two of the three Cartan generators J_i, $i = 1, 2, 3$, of the SU(4) R-symmetry. On the string theory side, we are considering type IIB string theory on AdS$_5 \times S^5$, E is the energy of a string state while $J = J_1 + J_2$ is the sum of two of the three Cartan generators J_i, $i = 1, 2, 3$, of the SO(6) symmetry of the five-sphere, all measured in units of the five-sphere radius. Moreover, $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling of SU(N) $\mathcal{N} = 4$ SYM, which on the string theory side is mapped to $R^4/(\alpha')^2$, $\sqrt{\alpha'}$ being the string length and R the radius of AdS$_5$ and S^5.

The leading part of the dynamics in the regime (1.1) corresponds to the decoupled theory that one obtains by taking the following decoupling limit [5–7][1]

$$\lambda \to 0, \quad J_i, \ N \text{ fixed}, \quad H \equiv \frac{E - J}{\lambda} \text{ fixed} \tag{1.2}$$

On the gauge theory side, we have in the planar limit $N = \infty$ that H is the Hamiltonian of a ferromagnetic $XXX_{1/2}$ Heisenberg spin chain with the single-trace operators interpreted as states of the spin chain [9, 5]. An important ingredient in this is that only states in the SU(2) sector can survive the limit. These are the states built only of the two scalars of $\mathcal{N} = 4$ SYM with $J = 1$. For all other states of $\mathcal{N} = 4$ SYM it is easy to see that $E - J$ becomes at least of order one, thus H goes to infinity in the above limit (1.2).

For $J \gg 1$ the Landau-Lifshitz sigma-model plus higher derivative terms gives an effective long wave-length description of the Heisenberg spin chain [10]. Using this we observe that we can find semi-classical states on the gauge theory side, $i.e.$ gauge theory states that have a large value for the sigma-model action when J is large.[2] This could seem surprising in that we are in weakly coupled gauge theory $\lambda \ll 1$, $i.e.$ it contradicts the standard lore that one should only find semi-classical string states for $\lambda \gg 1$. This observation motivates us to show that the regime (1.1) can be a semi-classically valid regime for strings on AdS$_5 \times S^5$ even though the effective string tension is small $R^2/\alpha' \ll 1$, which normally would mean that we are deep into a quantum string regime.

On the string side the limit (1.2) can be written as

$$\frac{R^2}{\alpha'} \to 0, \quad J \text{ fixed}, \quad H \equiv \frac{(\alpha')^2}{R^4}(E - J) \text{ fixed}, \quad \tilde{g}_s \equiv g_s \frac{(\alpha')^2}{R^4} \text{ fixed} \tag{1.3}$$

This limit involves taking the effective string tension R^2/α' to zero, again suggesting that we are deep into a quantum string regime. However, we find that when taking the limit (1.3) on the sigma-model for AdS$_5 \times S^5$ the action for the surviving string modes remains finite and is moreover large when J is large. $I.e.$ writing schematically the bosonic sigma-model

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[1]This decoupling limit was originally conceived as a limit in the Grand Canonical ensemble in which you are close to a critical point with zero temperature and critical chemical potential [5, 6]. A closely related limit has been considered in [8] corresponding to putting an extra chemical potential in the decoupled theory. In [7] limits giving other sectors than the SU(2) sector have been found.

[2]We find semi-classical string states when $\lambda/(E - J) \sim J$ with $\lambda \ll 1$ and $J \gg 1$, thus we are in the regime (1.1).

for $AdS_5 \times S^5$ as

$$I = -\frac{R^2}{4\pi\alpha'} \int d^2\sigma G_{\mu\nu}\partial^\alpha X^\mu \partial_\alpha X^\nu \qquad (1.4)$$

we find that the action remains finite in the limit (1.3) due to the fact that $\int d^2\sigma G_{\mu\nu}\partial^\alpha X^\mu \partial_\alpha X^\nu$ scales like $J\alpha'/R^2$ in the limit (1.3), thus making J the effective string tension in the regime (1.1). The regime (1.1) is therefore a new semi-classical regime of type IIB string theory on $AdS_5 \times S^5$. This is in agreement with the gauge theory side where we also find semi-classical string states in the regime (1.1).

Taking the limit (1.3) on the level of the classical string theory sigma-model we end up with the Landau-Lifshitz sigma-model. This resembles a similar limit of the classical bosonic sigma-model on $\mathbb{R} \times S^3$ considered by Kruczenski [11].[3] We consider subsequently the possible quantum corrections to the string theory sigma-model that can contribute in the limit (1.3). One reason that our analysis holds is due to the exactness of the supersymmetric string action on $AdS_5 \times S^5$ [12–15]. Another important aspect is the decoupling of six transverse bosonic fields, plus all the fermionic fields, which plays a crucial role. We argue that these modes become infinitely heavy and thus decouple in the limit (1.3) and through integrating them out they can only show up as higher-derivative terms for the surviving modes. In addition, we argue that zero-mode quantum effects for the decoupled modes are absent since we are close to $E = J$ which corresponds to half-BPS supersymmetric states.

By analyzing the classical sigma-model and the quantum effects, we conclude that the limit (1.3) gives the Landau-Lifshitz sigma-model up to $1/J^2$ corrections where the quantum effects can set in. The quantum effects enters as higher derivative terms coming from integrating out the decoupled modes. We can therefore match the effective sigma-model action for the strings, up to order $1/J^2$ corrections, to the sigma-model action obtained on the gauge theory side by considering large J. This enables us furthermore to show that not only we can match the leading order energy of semi-classical states but also the energy of quantum string states, up to $1/J^2$ corrections.

It might seem like we have found a string/gauge-theory duality which is a weak-weak duality. However, this is not the case. Instead, what we see on the gauge theory side is that the effective coupling is not λ but rather $\lambda/(E-J)$ when taking the limit (1.2). Thus, for the gauge theory, (1.1) is really a strong coupling regime since $\lambda/(E-J) \gg 1$. However, differently from usual, we have complete control over this regime by only knowing the one-loop contribution to the anomalous dimension of gauge theory operators.[4] Moreover, the

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[3]The way we take the limit (1.3) of the classical sigma-model on $\mathbb{R} \times S^3$ resembles closely the limit of Kruczenski [11]. However, the limit is not the same as the one considered by Kruczenski. The most important difference is that we do not assume we are in the semi-classical regime $R^2/\alpha' \gg 1$ in our limit. This is connected to the fact that we consider closely how the quantum effects come into play in our limit. It is also important to remark that the way we take our limit of the sigma-model is completely determined from the limit (1.2).

[4]As pointed out to us by Erik Verlinde, our limit has similarities with the 't Hooft limit where $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed. Here g_{YM}^2 is sent to zero in the limit but $\lambda \gg 1$ is still a strong coupling regime. Moreover, in the 't Hooft limit you access a simpler strong coupling regime than in the finite N theory (since you only have planar diagrams) which is somewhat analogous to the situation in our limit where we

identification of the one-loop dilatation operator as a spin chain is crucial for understanding the spectrum of the gauge theory side in the regime (1.1). Therefore, in this sense it should not be surprising that the regime (1.1) is under control in weakly coupled string theory, in that it corresponds to a particular kind of strong coupling regime of the gauge theory side.

In previous work on matching gauge theory and string theory in the AdS/CFT correspondence the starting point is that one should connect the weakly coupled gauge theory regime $\lambda \ll 1$ with the semi-classical string theory regime $\lambda \gg 1$. In particular, for gauge theory and string theory states in the SU(2) sector one can on both sides of the AdS/CFT correspondence make the expansion in $\lambda' = \lambda/J^2$ of the energy as follows

$$E - J = \lambda' E_1 + \lambda'^2 E_2 + \cdots \tag{1.5}$$

since J is large on both sides. In particular, it has been observed that at order λ', *i.e.* the one-loop contribution on the gauge theory side, you find the same energy from gauge theory and string theory up to $1/J^2$ corrections [9, 16 – 21], even though you compute it in two different regimes of the AdS/CFT correspondence. This matching of the energies begs for an explanation. Using our results we are able to provide this explanation by giving a simple argument for why one should obtain the same result for the λ' contribution for string theory in the regime $\lambda \gg 1$ as in the regime (1.1). This relies on our result that the effective sigma-model for type IIB string theory on AdS$_5 \times S^5$ in the limit (1.3) is given by the Landau-Lifshitz sigma-model up to $1/J^2$ corrections.

It is important to note that in the limit (1.2) we have $\lambda' = \lambda/J^2 \to 0$ hence this corresponds to taking a large volume limit with respect to the wrapping interactions for the spin chain description of $\mathcal{N} = 4$ SYM [22, 23], *i.e.* it is a limit in which wrapping effects are suppressed and the phase factor in the S-matrix description of the asymptotic Bethe equations for $\mathcal{N} = 4$ SYM is trivial [24 – 26]. Thus, our results show that one can match gauge theory and string theory in the AdS/CFT correspondence in this regime. This is consistent with the fact that the conjectured Bethe equations for quantum strings [24] become the Bethe equations for the Heisenberg sigma-model in the limit (1.2). Thus, the results of this paper provide an argument for why this should be the case.

We discuss furthermore the physical interpretation of our decoupling limit (1.2). On the string theory side, we show that the limit (1.3) (or equivalently (1.2)) in fact is a non-relativistic limit for type IIB string theory on AdS$_5 \times S^5$. *I.e.* it is a low energy limit and a limit of slow velocities for the strings. Moreover, we show that the decoupling of certain modes of the strings corresponds to going from a relativistic field theory, where we have an anti-particle for each particle, to a Galilean field theory. This is furthermore connected to the fact that we obtain a space-space non-commutative theory in the limit (1.3). We explain that this is because the effective sigma-model should describe a one-dimensional spin chain, hence the two spatial directions become the two directions in a phase space for a single spatial direction.

We consider briefly the interplay between the decoupling limit (1.2) and the Penrose limit of [27], which is a geometric limit of the AdS$_5 \times S^5$ background giving the maximally

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have full control over the strong coupling regime.

supersymmetric pp-wave background of [28], in the coordinate system with a flat direction [29, 27]. We explain that we can consider the two limits in different successions and that based on the results of this paper one finds the same limiting theory, which is a free theory with Galilean symmetry, regardless of the succession of the limits.

Finally, we consider the implications of the results of this paper for the matching of the Hagedorn temperature in [6] (see also [8]). We explain that the results of this paper puts the matching of the Hagedorn temperature on a firm basis since they show that one can match the leading order spectra of gauge theory and string theory in the limit (1.2) for large J. We can conclude from this that the Hagedorn temperature is the first example of a quantity not protected by supersymmetry that has been interpolated successfully from the weakly coupled gauge theory to the semi-classical string theory regime.

2. Decoupling limit of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

In this section we review briefly the decoupling limit of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ [5–7] giving a decoupled theory with an SU(2) symmetry.[5] We consider it here in terms of the charges and the energy/scaling dimension, $i.e.$ in the microcanonical ensemble. We review in particular that the decoupled theory in the planar limit corresponds to the ferromagnetic Heisenberg $XXX_{1/2}$ spin chain, and that one can take a continuum limit in which we can approximate the Heisenberg spin chain by a sigma-model.

Note that we discuss in section 7 and in the conclusions in section 8 why $\lambda/(E-J) \gg 1$ can be seen as a strong coupling regime.

2.1 Review of decoupling limit

We review here briefly the SU(2) decoupling limit of [5–7] on the gauge theory side of the AdS/CFT correspondence. We are considering SU(N) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Since we take the large N limit below we introduce the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$. We denote the three R-charges for the SO(6) $\simeq$ SU(4) R-symmetry as J_1, J_2, J_3. We employ in the following the state/operator correspondence relating a state of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ of energy E to an operator of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ of scaling dimension $D = E$, $i.e.$ we set the radius of the S^3 to one. Due to the compactification on S^3 the states are restricted to be singlets of SU(N) which on the operator side restricts us to the class of operators consisting of linear combinations of multi-trace operators.

The SU(2) decoupling limit of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. We consider the following decoupling limit of SU(N) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ [5–7]

$$\lambda \to 0, \quad J_i, \, N \text{ fixed}, \quad H \equiv \frac{E - J}{\lambda} \text{ fixed} \tag{2.1}$$

with $J \equiv J_1 + J_2$ and $\lambda = g_{\text{YM}}^2 N$. Note in particular that J is fixed in the limit. In terms of operators we have that $H = (D - J)/\lambda$. The scaling dimension is found by diagonalizing

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[5]There are altogether 12 non-trivial decoupled theories [7]. These 12 theories correspond to twelve different classes of limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. In the grand canonical ensemble they correspond to being close to either of the twelve different critical points.

the dilatation operator D [30, 16, 31]. At weak 't Hooft coupling we expand D as

$$D = D_0 + \lambda D_2 + \lambda^{\frac{3}{2}} D_3 + \lambda^2 D_4 + \cdots \tag{2.2}$$

Here D_0 is the bare scaling dimension and D_2 is the one-loop part of the dilatation operator (see [32] for a complete expression). Taking now the limit (2.1) we see that since $D_0 - J$ is a integer or half-integer we have that $(D - J)/\lambda$ goes to infinity in the limit unless $D_0 = J$. Thus, all operators with $D_0 > J$ decouple in the limit. Note here that any operator in $\mathcal{N} = 4$ SYM obeys the bound $D_0 \geq J$. The class of operators saturating the bound, $i.e.$ that has $D_0 = J$, corresponds to the so-called SU(2) sector of $\mathcal{N} = 4$ SYM consisting of all possible operators that one can make from linear combinations of multi-trace operators built from the single-trace operators of the form

$$\mathrm{Tr}(A_1 A_2 \cdots A_J), \quad A_i \in \{Z, X\} \tag{2.3}$$

Here Z and X are two of the three complex scalars of $\mathcal{N} = 4$ SYM with R-charges $(J_1, J_2, J_3) = (1, 0, 0)$ for Z and $(J_1, J_2, J_3) = (0, 1, 0)$ for X. Since $J = J_1 + J_2$ we see that the total number of Z's and X's add up to J for any operator.[6]

For states in the SU(2) sector we see now that from $H = (D - J)/\lambda$ we get an effective Hamiltonian

$$H = D_2 \tag{2.4}$$

in the limit (2.1). This is a Hamiltonian in the sense that we get the energies/scaling dimensions of the surviving states/operators by diagonalizing H. We have explicitly [33, 16]

$$H = -\frac{\lambda}{8\pi^2 N} \mathrm{Tr}[X, Z][\bar{X}, \bar{Z}] \tag{2.5}$$

where $\bar{X} = \delta/\delta X$ and $\bar{Z} = \delta/\delta Z$. This Hamiltonian gives the spectrum of our decoupled theory.

Planar limit corresponds to ferromagnetic Heisenberg chain. Considering now the planar limit $N = \infty$ we can employ large N factorization and get the scaling dimension of any operator from knowing the scaling dimension of single-trace operators. Also, by the same token the mixing between single-trace operators and multi-trace operators goes to zero. Thus, we can get the whole spectrum by just focusing on the single-trace operators. Considering now the single-trace operators in the SU(2), we have that a single-trace operator (2.3) can be interpreted as a state in a spin 1/2 spin chain [9] with the letters Z and X being the spin up and spin down state.

In detail we choose the spin as $S_z = (J_1 - J_2)/2$ which means that $S_z = 1/2$ for Z and $S_z = -1/2$ for X. Thus, for each site on the spin chain we have a two-dimensional vector-space spanned by the spin-up and spin-down states. On this two-dimensional space we can define the spin vector $\vec{S}_i$ for site number i as $\frac{1}{2}\vec{\sigma}$ acting on the state of the i'th site, where $\vec{\sigma}$ are the Pauli matrices. In this way we see that $S_z = \sigma_z/2$ which is consistent with

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[6]Note that we keep all three R-charges fixed in the limit (2.1) thus we should take the limit with $J_3 = 0$ since if $J_3 \neq 0$ we decouple all operators.

the above definition of S_z. The Hamiltonian H, as defined by the limit (1.2), is then given by

$$H = \frac{1}{4\pi^2} \sum_{i=1}^{J} \left(\frac{1}{4} - \vec{S}_i \cdot \vec{S}_{i+1} \right) \tag{2.6}$$

This is the Hamiltonian for the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain with zero magnetic field [9]. Here J is the length of the spin chain and $S_z = (J_1 - J_2)/2$ is the total spin. Thus, in conclusion, this is the decoupled theory that one gets from taking the decoupling limit (2.1) of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$.

Since the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain is an integrable system we can write down the following equations that in principle determines the full spectrum of H [7]

$$H = \frac{1}{2\pi^2} \sum_{i=1}^{M} \sin^2 \left(\frac{p_i}{2} \right) \tag{2.7}$$

$$e^{ip_k J} = \prod_{j=1, j \neq k}^{M} S(p_k, p_j), \quad S(p_k, p_j) = -\frac{1 + e^{i(p_k + p_j)} - 2e^{ip_k}}{1 + e^{i(p_k + p_j)} - 2e^{ip_j}}, \quad \sum_{i=1}^{M} p_i = 0 \tag{2.8}$$

These equations are the dispersion relation for H, the Bethe equations along with the S-matrix, and a zero total momentum condition due to the cyclicity of the trace. We have introduced here M momenta p_i corresponding to M magnons which are pseudoparticles propagating on the chain.

For J large, we can consider the low energy part of the spectrum $H \ll 1$. This corresponds to having the momenta of the magnons of order $1/J$ to leading order. One then finds from (2.7)-(2.8) the following leading order low energy spectrum of H

$$H = \frac{1}{2J^2} \sum_{n \neq 0} \left(1 + \frac{2}{J} \right) n^2 M_n + \mathcal{O}(1/J^2), \quad \sum_{n \neq 0} n M_n = 0 \tag{2.9}$$

where M_n is the number operator for the integer level n with $n \neq 0$. Note that this spectrum is only true for states built from magnons with different momenta p_i, *i.e.* it is not true for bound states. We ignore this subtlety for simplicity of presentation.

We see from the spectrum (2.9) that for large J the Hamiltonian H goes like $1/J^2$ for the low energy excitations. It is therefore natural in this regime to introduce the rescaled Hamiltonian $\tilde{H}$

$$\tilde{H} = J^2 H = \frac{J^2}{\lambda}(E - J) = \frac{J^2}{4\pi^2} \sum_{i=1}^{J} \left(\frac{1}{4} - \vec{S}_i \cdot \vec{S}_{i+1} \right) \tag{2.10}$$

such that the spectrum (2.9) is

$$\tilde{H} = \frac{1}{2} \sum_{n \neq 0} \left(1 + \frac{2}{J} \right) n^2 M_n + \mathcal{O}(1/J^2), \quad \sum_{n \neq 0} n M_n = 0 \tag{2.11}$$

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[7]See [6] for a construction of the Bethe ansatz that takes into account that the total spin $S_z = (J_1 - J_2)/2$ is fixed in our decoupled theory.

We see that focusing on the low energy part of the spectrum of H in the large J limit corresponds to considering the part of the spectrum of $\tilde{H}$ which is of order one.

Note that $\tilde{H} \sim 1$ means that $\lambda/(E - J) \sim J^2$. Thus, this falls within the regime (1.1). Therefore we can conclude that we find quantum string states in planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the regime (1.1).

2.2 Effective sigma-model description in continuum limit

In this section we review that the ferromagnetic Heisenberg spin chain, that we obtain from our decoupling limit (1.2) of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, has a sigma-model description for large J [10]. We review this briefly in order to compare to what happens on the string side in the decoupling limit (1.2) in section 3.

In the context of AdS/CFT, the sigma-model limit of the Heisenberg spin chain was first used in [11]. We comment below, and also in the rest of this paper, in what sense our approach is different from that of [11].

To obtain a sigma-model description of the Heisenberg spin chain, one begins by introducing a coherent state $|\vec{n}\rangle$ for each site of the spin chain such that

$$\langle \vec{n} | \vec{\sigma} | \vec{n} \rangle = \vec{n} \tag{2.12}$$

where $\vec{\sigma}$ are the two by two Pauli matrices. Here $\vec{n}$ is a unit vector pointing to a point on the two-sphere parameterized as

$$\vec{n} = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, \sin\theta) \tag{2.13}$$

One then proceeds to first write up the one-spin partition function, ignoring the interaction between different spins. This can be done using the usual derivation of the path-integral in quantum mechanics. One can then use this to write up the partition function for the full spin chain, now including the interaction Hamiltonian (2.10). Altogether, this gives the partition function and action [10]

$$Z = \int \mathcal{D}\vec{n} \, e^{iI[\vec{n}]}, \quad I[\vec{n}] = \sum_{k=1}^{J} \int d\tilde{t} \left[\vec{C}(\vec{n}_k) \cdot \partial_{\tilde{t}} \vec{n}_k - \frac{J^2}{32\pi^2} (\vec{n}_{k+1} - \vec{n}_k)^2 \right] \tag{2.14}$$

where

$$\vec{C}(\vec{n}) \cdot \partial_{\tilde{t}} \vec{n} = -\frac{1}{2} \int_0^1 d\xi \, \epsilon_{ijk} n_i \partial_\xi n_j \partial_{\tilde{t}} n_k = \frac{1}{2} \sin\theta \, \partial_{\tilde{t}} \varphi \tag{2.15}$$

is a Wess-Zumino type term where $\vec{C}(\vec{n})$ is proportional to the area spanned between the trajectory and the north pole of the two-sphere [10]. The action (2.14) provides an equivalent description of the Heisenberg spin chain. Notice that this action is describing a one-dimensional lattice of J spins, i.e. $\vec{n}_k$ is the spin on the k'th site of the lattice.

It is important to note that in deriving the action (2.14) we have used a time $\tilde{t}$ corresponding to $\tilde{H}$, i.e. $\tilde{H} = J^2(E - J)/\lambda = i\partial_{\tilde{t}}$. This is because we are interested in the low energy dynamics for J large, hence the regime in which $\tilde{H}$ is of order one.

Taking now the limit $J \to \infty$, we can approximate the one-dimensional lattice by a continuous variable. Denoting this variable σ we are considering the field $\vec{n}(\tilde{t}, \sigma)$. Therefore,

imposing that σ has period 2π, we should map the k'th site to $\sigma = 2\pi k/J$, *i.e.* $\vec{n}_k(\tilde{t})$ is mapped to $\vec{n}(\tilde{t}, \sigma)$. Correspondingly we map the sum $\sum_{k=1}^{J}$ to the integral $\frac{J}{2\pi} \int_0^{2\pi} d\sigma$. We furthermore use that

$$\vec{n}_{k+1} - \vec{n}_k = \exp\left(\frac{2\pi}{J}\partial_\sigma\right)\vec{n} - \vec{n} \tag{2.16}$$

This gives the action

$$I[\vec{n}] = \frac{J}{2\pi} \int d\tilde{t} \int_0^{2\pi} d\sigma \left[\vec{C}(\vec{n}) \cdot \partial_{\tilde{t}}\vec{n} + \frac{J^2}{8\pi^2}\vec{n} \cdot \sinh^2\left(\frac{\pi}{J}\partial_\sigma\right)\vec{n}\right] \tag{2.17}$$

We see that this can be considered to be a sigma-model on a continuous world-sheet parameterized by $\tilde{t}$ and σ, with the target space given by $S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$. The first term in (2.17) is a kinetic term, while the second one is a potential term. The second term is responsible for the dispersion relation (2.7). This can be seen from the fact that the momenta p of an impurity is mapped to $-i(2\pi/J)\partial_\sigma$. We see that the discreteness of the Heisenberg spin chain manifests itself as a infinite sum over higher derivative terms in the continuum action (2.17).

Expanding now the action (2.17) in powers of $1/J$ we have

$$I[\vec{n}] = \frac{J}{2\pi} \int d\tilde{t} \int_0^{2\pi} d\sigma \left[\vec{C}(\vec{n}) \cdot \dot{\vec{n}} - \frac{1}{8}(\vec{n}')^2 + \frac{\pi^2}{24J^2}(\vec{n}'')^2 + \mathcal{O}(J^{-4})\right] \tag{2.18}$$

where we introduced a dot (prime) as the derivative with respect to $\tilde{t}$ (σ). The leading part of the action (2.17) in the thermodynamic limit $J \to \infty$ is therefore

$$I[\theta, \varphi] \simeq I_{\mathrm{LL}}[\theta, \varphi] \equiv \frac{J}{4\pi} \int d\tilde{t} \int_0^{2\pi} d\sigma \left[\sin\theta\dot{\varphi} - \frac{1}{4}\left[(\theta')^2 + \cos^2\theta(\varphi')^2\right]\right] \tag{2.19}$$

here written in terms of the parametrization (2.13). We see that to leading order in the thermodynamic limit the Heisenberg spin chain is well-described by the Landau-Lifshitz model with action $I_{\mathrm{LL}}[\theta, \varphi]$.

Finally, we note that the constraint of zero total momentum $\sum_{i=1}^{M} p_i = 0$ in (2.8) takes the following form for the sigma-model $\int_0^{2\pi} d\sigma \sin\theta\varphi' = 0$ in terms of the parametrization (2.13).

Getting the spectrum from the sigma-model. We now explain briefly how to get the spectrum (2.11) from the sigma-model. We begin by considering a limit of the action (2.17), here dubbed the free limit, in which the sigma-model reduces to a free theory with Galilean symmetry. The free limit is a large J limit in which we zoom in to a point on the equator of the two-sphere. More specifically, it is a limit in which we zoom in near the point $(\theta, \varphi) = (0, 0)$ such that the two-sphere metric $d\Omega_2^2 \simeq d\theta^2 + d\varphi^2$, which means that the geometry near this point is 2D Euclidean space. We take the free limit by defining the rescaled coordinates

$$x = \sqrt{J}\varphi, \quad y = \sqrt{J}\theta \tag{2.20}$$

which we keep fixed as $J \to \infty$. Taking now the $J \to \infty$ limit of the action (2.19) we get the following action

$$I = \frac{1}{4\pi} \int d\tilde{t} \int_0^{2\pi} d\sigma \left[y\dot{x} - \frac{(x')^2 + (y')^2}{4}\right] \tag{2.21}$$

This action can easily be quantized. Define $z = x + iy$. Then we can write the EOMs as $\dot{z} = \frac{i}{2}z''$. Using the EOMs plus the periodicity of σ we see that the general expansion of $z(\tilde{t}, \sigma)$ is

$$z(\tilde{t}, \sigma) = 2\sum_{n \in \mathbb{Z}} a_n e^{-i\frac{n^2}{2}\tilde{t} + in\sigma} \tag{2.22}$$

To quantize the theory, we note that from the action (2.21) we have that the conjugate momentum to x is $p_x = y/(4\pi)$. The canonical commutation relation is

$$[x(\tilde{t}, \sigma), p_x(\tilde{t}, \sigma')] = i\delta(\sigma - \sigma') \tag{2.23}$$

Using this with (2.22) we see that the a_n's becomes lowering operators with the canonical commutation relation

$$[a_n, a_k^\dagger] = \delta_{nk} \tag{2.24}$$

We see from this that $z(\tilde{t}, \sigma)$ only contains lowering operators. This means that we do not have an anti-particle part of $z(\tilde{t}, \sigma)$. This fits with the fact that we have a non-relativistic dispersion relation $H \propto p_\sigma^2$ suggesting that we do not have an antiparticle propagating backwards in time like in relativistic field theory.[8] We remark furthermore that since we have that $y = 4\pi p_x$ the two transverse dimensions that we started with have become the two-dimensional phase space for the one dimension x. Thus we see a reduction from two space-like dimensions to just one space-like dimension. This is connected to the non-relativistic nature of the action (2.21) since normally two spatial directions in a sigma-model would give rise the double number of raising and lowering operators as we found above. We consider further the non-relativistic nature of the Heisenberg model in section 5.

From the action (2.21) we see that the Hamiltonian is

$$\tilde{H} = \int_0^{2\pi} d\sigma \left[\frac{(x')^2}{16\pi} + \pi(p_x')^2 \right] \tag{2.25}$$

Inserting now (2.22) in this, we get the spectrum

$$\tilde{H} = \frac{1}{2}\sum_{n \in \mathbb{Z}} n^2 M_n \,, \quad \sum_{n \in \mathbb{Z}} n M_n = 0 \tag{2.26}$$

with the number operator being $M_n = a_n^\dagger a_n$. This spectrum matches the leading order part of the spectrum (2.11). The second equation is the level-matching condition derived by imposing the vanishing of the total world-sheet momentum.[9]

One can also find easily the $1/J$ correction to the leading spectrum (2.11) from the sigma-model (see for example [35]). We begin using the coordinate x as defined in (2.20). The conjugate momentum p_x for the full action (3.23) is

$$p_x = \frac{\sqrt{J}}{4\pi}\sin\theta \tag{2.27}$$

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[8]The non-relativistic nature of the Landau-Lifshitz sigma-model is also considered in [34].

[9]This is derived from $\int_0^{2\pi} d\sigma y x' = 0$.

Inserting this in the action (3.23), we can easily derive the Hamiltonian

$$\tilde{H} = \int_0^{2\pi} d\sigma \left[\left(1 - \frac{16\pi^2}{J} p_x^2 \right) \frac{(x')^2}{16\pi} + \left(1 + \frac{16\pi^2}{J} p_x^2 \right) \pi (p_x')^2 + \mathcal{O}(J^{-2}) \right] \qquad (2.28)$$

To find the $1/J$ corrections one can use ordinary quantum mechanical perturbation theory and plug in the zeroth order x and p_x, as found from (2.22), into the Hamiltonian (2.28). Doing this, one obtains precisely the corrected spectrum (2.11).[10]

Semi-classical states in decoupled theory. The spectrum (2.11) which has $\tilde{H}$ of order one, corresponds to considering a finite number of impurities for the Bethe equations (2.7)-(2.8). If we instead consider a number of impurities M such that M/J is of order one, we get that the energy $\tilde{H}$ of such states is of order J. Such states are semi-classical since when considering large quantum numbers we can approximate the quantum physics with classical physics.

From the sigma-model point of view we have a natural classical description of states with M/J of order one. These are the classical solutions of the Landau-Lifshitz sigma-model (2.19). It is clear that any finite size solution of the sigma-model (2.19), *i.e.* solutions that extend out in a finite area on the two-sphere, will correspond to an energy $\tilde{H}$ of order J since the action (2.19) is proportional to J. Therefore we see that we can find semi-classical solutions of the sigma-model in the decoupled theory. Note also that it is clear from (2.19) that a finite-size solution on the two-sphere can be well-described classically since the action (2.19) will be large when J is large.

In conclusion we have that semi-classical string states appears for $\tilde{H} \sim J$, *i.e.* for $\lambda/(E - J) \sim J$. Therefore, we find semi-classical string states in planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the regime (1.1).

3. Decoupling limit of strings on AdS$_5 \times S^5$

In this section we implement the decoupling limit (1.2) on type IIB string theory on AdS$_5 \times S^5$. This is accomplished by first considering the limit on a purely classical level. In this way we obtain the Landau-Lifshitz model as the limiting sigma-model. Subsequently we consider the quantum effects for the decoupling limit. We show that the transverse modes decouple and we argue why the quantum effects are under control in our limit (1.2) even though one naively seems to enter a quantum string regime. Finally we argue that this means that we can match the leading spectra (2.11) of gauge and string theory by taking the limit (1.2) on both sides of the AdS/CFT correspondence. This furthermore includes the semi-classical states for which the action is large on both the gauge theory and the string theory sides.

3.1 Classical limit of AdS$_5 \times S^5$ sigma-model

In the following we take the limit (1.2) of the classical sigma-model for AdS$_5 \times S^5$. The quantum effects are considered in section 3.2.

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[10]Note again that the spectrum (2.11) only describes non-bound-states, *i.e.* states build from raising operators with different levels. The leading part of the spectrum (2.26) is instead true for all states.

Note that the classical limit of the sigma-model considered in the following closely resembles the limit of Kruczenski in [11]. However, even though these limits resemble each other on the level of the classical sigma-model, they are different for the quantum string theory since Kruczenski takes $J \to \infty$ keeping λ/J^2 fixed whereas we take $\lambda \to 0$ keeping J fixed.

We are considering type IIB string theory on the $\text{AdS}_5 \times S^5$ background with metric

$$ds^2 = R^2 \left[-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, (d\Omega_3')^2 + d\zeta^2 + \sin^2 \zeta \, d\alpha^2 + \cos^2 \zeta \, (d\Omega_3)^2 \right] \quad (3.1)$$

and the five-form Ramond-Ramond field strength

$$F_{(5)} = 2R^4 \left[\cosh \rho \, \sinh^3 \rho \, dt \, d\rho \, d\Omega_3' + \sin \zeta \, \cos^3 \zeta \, d\zeta \, d\alpha \, d\Omega_3 \right] \quad (3.2)$$

We use in the following that

$$R^4 = \lambda(\alpha')^2 \quad (3.3)$$

This relates the string parameters R and α' to the 't Hooft coupling λ of $\mathcal{N} = 4$ SYM. Using this the limit (1.2) can be formulated in string theory variables as (1.3). However, we choose below instead to use the variables of the gauge theory.

We parameterize the three-sphere Ω_3 as

$$(d\Omega_3)^2 = d\psi^2 + \cos^2 \psi d\phi_1^2 + \sin^2 \psi d\phi_2^2 = d\psi^2 + d\phi_-^2 + d\phi_+^2 + 2\cos(2\psi)d\phi_- d\phi_+ \quad (3.4)$$

where $2\phi_\pm = \phi_1 \pm \phi_2$. The energy E and the SO(6) Cartan generators J_i, $i = 1, 2, 3$, are given by

$$E = i\partial_t, \quad J \equiv J_1 + J_2 = -i\partial_{\phi_+}, \quad S_z \equiv \frac{J_1 - J_2}{2} = -\frac{i}{2}\partial_{\phi_-}, \quad J_3 = -i\partial_\alpha \quad (3.5)$$

In the limit (1.2) we only consider the charges E, J_1 and J_2. Together with the fact that on the gauge theory side we decouple everything but the SU(2) sector in the limit (1.2) it seems evident that we can work in the region $\rho = \zeta = 0$ of the $\text{AdS}_5 \times S^5$ background (3.1)-(3.2). In this region the background is simply given by the metric $ds^2 = R^2[-dt^2 + (d\Omega_3)^2]$. Thus, we take the bosonic sigma-model for $\mathbb{R} \times S^3$ to be the starting point below. This is obviously only valid classically since we can have quantum fluctions in the directions transverse to $\rho = \zeta = 0$ (along with fermionic fluctuations). We deal with these issues in section 3.2.

We take now as starting point the $\mathbb{R} \times S^3$ background $ds^2 = R^2[-dt^2 + (d\Omega_3)^2]$ with $(d\Omega_3)^2$ given by (3.4). Define

$$\theta \equiv 2\psi - \frac{\pi}{2}, \quad \varphi \equiv 2\phi_- \quad (3.6)$$

then we have the metric

$$ds^2 = R^2 \left[-dt^2 + \frac{1}{4}(d\Omega_2)^2 + \left(d\phi_+ + \frac{1}{2}\sin\theta d\varphi \right)^2 \right] \quad (3.7)$$

with the two-sphere metric given as

$$(d\Omega_2)^2 = d\theta^2 + \cos^2\theta d\varphi^2 \tag{3.8}$$

To approach the right energy scale, we make the coordinate transformation

$$\tilde{t} = \frac{\lambda}{J^2}t\,, \quad \chi = \phi_+ - t \tag{3.9}$$

This ensures that $\tilde{H} \equiv (E - J)J^2/\lambda = i\partial_{\tilde{t}}$ which precisely corresponds to the energy that we found was relevant in the sigma-model description on the gauge theory side (2.10). Moreover, we have that $J = -i\partial_\chi$ and $S_z = -i\partial_\varphi$. With this the metric (3.7) is

$$ds^2 = \sqrt{\lambda}\,\alpha'\left[\frac{J^2}{\lambda}d\tilde{t}\Big(2d\chi + \sin\theta d\varphi\Big) + \frac{1}{4}(d\Omega_2)^2 + \Big(d\chi + \frac{1}{2}\sin\theta d\varphi\Big)^2\right] \tag{3.10}$$

Consider now the sigma-model Lagrangian

$$\mathcal{L} = -\frac{1}{2}G_{\mu\nu}h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \tag{3.11}$$

We pick the gauge

$$\tilde{t} = \kappa\tau\,, \quad p_\chi = \text{const.}\,, \quad h_{\alpha\beta} = \eta_{\alpha\beta} \tag{3.12}$$

with $p_\chi \equiv \partial\mathcal{L}/\partial\partial_\tau\chi$. Employing this, the Lagrangian (3.11) is found to be

$$\mathcal{L} = \frac{\sqrt{\lambda}\alpha'}{2}\left[\kappa\frac{J^2}{\lambda}\Big(2\partial_\tau\chi + \sin\theta\partial_\tau\varphi\Big) + \frac{1}{4}\Big((\partial_\tau\theta)^2 + \cos^2\theta(\partial_\tau\varphi)^2 - (\theta')^2 - \cos^2\theta(\varphi')^2\Big)\right.$$
$$\left. + \Big(\partial_\tau\chi + \frac{1}{2}\sin\theta\partial_\tau\varphi\Big)^2 - \Big(\chi' + \frac{1}{2}\sin\theta\varphi'\Big)^2\right] \tag{3.13}$$

The Virasoro constraints are $G_{\mu\nu}\partial_\tau x^\mu\partial_\sigma x^\nu = 0$ and $G_{\mu\nu}(\partial_\tau x^\mu\partial_\tau x^\nu + \partial_\sigma x^\mu\partial_\sigma x^\nu) = 0$, giving

$$0 = \sqrt{\lambda}\,\alpha'\left[\kappa\frac{J^2}{\lambda}\Big(\chi' + \frac{1}{2}\sin\theta\varphi'\Big) + \frac{1}{4}\Big(\partial_\tau\theta\theta' + \cos\theta\partial_\tau\varphi\varphi'\Big)\right.$$
$$\left. + \Big(\partial_\tau\chi + \frac{1}{2}\sin\theta\partial_\tau\varphi\Big)\Big(\chi' + \frac{1}{2}\sin\theta\varphi'\Big)\right] \tag{3.14}$$

$$0 = \kappa\frac{J^2}{\lambda}\Big(2\partial_\tau\chi + \sin\theta\partial_\tau\varphi\Big) + \frac{1}{4}\Big((\partial_\tau\theta)^2 + \cos^2\theta(\partial_\tau\varphi)^2 + (\theta')^2 + \cos^2\theta(\varphi')^2\Big)$$
$$+ \Big(\partial_\tau\chi + \frac{1}{2}\sin\theta\partial_\tau\varphi\Big)^2 + \Big(\chi' + \frac{1}{2}\sin\theta\varphi'\Big)^2 \tag{3.15}$$

We record that

$$p_\chi \equiv \frac{1}{2\pi\alpha'}\frac{\partial\mathcal{L}}{\partial\partial_\tau\chi} = \frac{\kappa}{2\pi}\frac{J^2}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{2\pi}\Big(\partial_\tau\chi + \frac{1}{2}\sin\theta\partial_\tau\phi\Big) \tag{3.16}$$

Note that the last term on the right-hand side goes away in the $\lambda \to 0$ limit which means that the result for p_χ is consistent with the above gauge choice (3.12). From this we see furthermore that

$$J = \int_0^{2\pi} d\sigma p_\chi = \kappa\frac{J^2}{\sqrt{\lambda}} \tag{3.17}$$

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This means that we have

$$\kappa = \frac{\sqrt{\lambda}}{J} \tag{3.18}$$

We take now the $\lambda \to 0$ limit of the Lagrangian and the constraints. For the Lagrangian we get

$$\frac{1}{\kappa}\mathcal{L} = \frac{\alpha' J}{2}\left[\left(2\dot{\chi} + \sin\theta\dot{\varphi}\right) - \frac{1}{4}\left((\theta')^2 + \cos^2\theta(\varphi')^2\right) - \left(\chi' + \frac{1}{2}\sin\theta\varphi'\right)^2\right] \tag{3.19}$$

where we defined the dot as a derivative with respect to $\tilde{t}$. For the constraints we get

$$\chi' = -\frac{1}{2}\sin\theta\varphi', \quad \dot{\chi} = -\frac{1}{2}\sin\theta\dot{\varphi} - \frac{1}{8}\left((\theta')^2 + \cos^2\theta(\varphi')^2\right) \tag{3.20}$$

We can now eliminate χ from the Lagrangian. Using the first constraint we see that we only have a $\dot{\chi}$ term in the Lagrangian without coupling to the other fields. Because of this, we can ignore it in the Lagrangian, since omitting this term do not affect the EOMs for the other fields. We can thus write the gauge fixed Lagrangian

$$\frac{1}{\kappa}\mathcal{L}_{\text{gf}} = \frac{\alpha' J}{2}\left[\sin\theta\dot{\varphi} - \frac{1}{4}\left((\theta')^2 + \cos^2\theta(\varphi')^2\right)\right] \tag{3.21}$$

This is then supplemented with the two constraints (3.20) that determine χ from the other fields. Writing the action for the gauge-fixed Lagrangian $\mathcal{L}_{\text{gf}}$ as

$$I = \frac{1}{2\pi\alpha'}\int d\tau \int_0^{2\pi} d\sigma \mathcal{L}_{\text{gf}} = \frac{1}{2\pi\alpha'}\int d\tilde{t} \int_0^{2\pi} d\sigma \frac{1}{\kappa}\mathcal{L}_{\text{gf}} \tag{3.22}$$

we see that the action of the resulting effective sigma-model after the $\lambda \to 0$ limit is

$$I = \frac{J}{4\pi}\int d\tilde{t} \int_0^{2\pi} d\sigma \left[\sin\theta\dot{\varphi} - \frac{1}{4}\left((\theta')^2 + \cos^2\theta(\varphi')^2\right)\right] \tag{3.23}$$

From the first constraint in (3.20) we see that the action (3.23) should be supplemented by the condition that the total world-sheet momentum is zero

$$\int_0^{2\pi} d\sigma \sin\theta\varphi' = 0 \tag{3.24}$$

We see that (3.23) precisely corresponds to the leading order part (2.19) of the sigma-model action (2.17) derived on the gauge theory side. Thus, also on the string theory side we regain the Landau-Lifshitz model. This is encouraging since we are getting the same action on the gauge theory and string theory sides of AdS/CFT by taking the same limit on both sides of the correspondence. However on the string side our limit is taken, so far, purely classically. This is, as we discuss below, also the reason why we get the Landau-Lifshitz model exactly in (3.23) whereas on the gauge theory side the leading order action (2.17) is only an approximation. This point will be resolved in section 3.2.

It is important to note that if we consider solutions of the sigma-model (3.23) which are of finite-size on the two-sphere then the action (3.23) is large when J is large. Therefore, already at this point we see that we can safely match semi-classical states on the string theory side to semi-classical states on the gauge theory side when J is large. We comment further on this below.

3.2 Taking into account quantum effects

In section 3.1 we took the decoupling limit (1.2) of type IIB string theory on $\text{AdS}_5 \times S^5$ on the level of the classical sigma-model. In the following we consider the quantum effects to see how the limit (1.2) works in the quantized string theory.

In section 3.1 we saw that taking the limit (1.2) classically gives the Landau-Lifshitz sigma-model (3.23) without assuming J large. As noted above, this is a problem since on the gauge theory side the Landau-Lifshitz sigma-model is only valid for large J (2.19). This problem will be resolved in the following by taking into account the quantum effects.

In the AdS/CFT correspondence we have that planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is dual to tree-level string theory on the $\text{AdS}_5 \times S^5$ background (3.1)-(3.2). Tree-level string theory means that we are considering first-quantized strings on $\text{AdS}_5 \times S^5$. We can write schematically the full partition function for first-quantized type IIB strings on the $\text{AdS}_5 \times S^5$ background (3.1)-(3.2) as

$$Z = \int [Dh][Dx][DS]e^{iI[h,x,S]} \tag{3.25}$$

where h is the world-sheet metric, x the bosonic fields and S the fermionic fields. We have from the background (3.1)-(3.2) that the action $I[h,x,S]$ is proportional to $R^2/\alpha' = \sqrt{\lambda}$, assuming we keep fixed the fields and the world-sheet metric. Therefore, it is customary to regard $\sqrt{\lambda}$ as an effective string tension on the $\text{AdS}_5 \times S^5$ background.

Now, in the decoupling limit (1.2) we take $\lambda \to 0$ as part of the limit. It therefore looks like we enter deep into the quantum string regime, since naively it seems that $I[h,x,S] \to 0$. However, this is not the case. As can be seen from the classical limit in section 3.1, the modes with energies $E - J$ of order λ in the limit (1.2) give a finite contribution to $I(h,x,S)$. This can be seen from the fact that the modes for which the limiting action (3.23) is finite also give a finite value to the full action $I(h,x,S)$ in the $\lambda \to 0$ limit. Thus, for these modes we can hope to have the quantum effects under control even though $\lambda \to 0$. We should be careful however because even though some modes give a finite contribution to $I(h,x,S)$ there can be significant changes to the action due to quantum effects.

Corrections to sigma-model action. One possible source of change of the action is that in general the target space background of a sigma-model receives α' corrections when imposing conformal invariance of the sigma-model. If such corrections occur they could significantly change how the sigma-model looks since, effectively speaking, we are taking $\alpha' \to \infty$. However, the $\text{AdS}_5 \times S^5$ background is known to be an exact background due to the large amount of supersymmetry [12], thus we can trust the sigma-model in our limit. In fact, in [13–15] exact gauge-fixed Lagrangians for the Green-Schwarz superstring on the $\text{AdS}_5 \times S^5$ background are found.

Decoupling of transverse modes. Another possible source of change of the action, which is more difficult to address, are the modes that do not give a finite contribution to $I(h,x,S)$. One set of such modes is the bosonic modes that correspond to fluctuations transverse to $\rho = \zeta = 0$. To understand these modes we rewrite the full metric (3.1) for

AdS$_5 \times S^5$ as

$$ds^2 = \cos^2 \zeta \, R^2 [-dt^2 + (d\Omega_3)^2] - R^2 (\sinh^2 \rho + \sin^2 \zeta) dt^2 + R^2 A_{ij} dx^i dx^j \qquad (3.26)$$

where x^i are the remaining directions (ρ, ζ, α and Ω_3') and $R^2 A_{ij}$ is the metric for these directions. We see from this that setting $\rho = \zeta = 0$ in (3.26) we end up with the metric on $\mathbb{R} \times S^3$ which is the starting point of our classical analysis in section 3.1.

Using now (3.26), we can write the full bosonic sigma-model Lagrangian for AdS$_5 \times S^5$ as

$$\frac{1}{\kappa} \mathcal{L}_{\text{full}} = \cos^2 \zeta \frac{1}{\kappa} \mathcal{L} + \frac{\sqrt{\lambda}}{2J} A_{ij} \dot{x}^i \dot{x}^j - \frac{J}{2\sqrt{\lambda}} A_{ij} x'^i x'^j - \frac{\alpha' J^3}{2\lambda} (\sinh^2 \rho + \sin^2 \zeta) \qquad (3.27)$$

We see that the term without derivatives in this Lagrangian corresponds to the potential term

$$\frac{J^3}{4\pi\lambda} (\sinh^2 \rho + \sin^2 \zeta) \qquad (3.28)$$

This is a confining potential. For $\lambda \to 0$ any mode with $\rho > 0$ or $\zeta > 0$ will be driven towards the origin $\rho = \zeta = 0$ by the confining potential. I.e. if we excite a mode so that $\rho > 0$ or $\zeta > 0$ then the energy of such a mode would be proportional to $1/\lambda$ which means that for $\lambda \to 0$ it would cost an infinite amount of energy to make such an excitation. Therefore we get that only modes with $\rho = \zeta = 0$ survive the limit (1.2).

Notice that the decoupling of modes transverse to $\rho = \zeta = 0$ is the string equivalent of the decoupling of the modes not in the SU(2) sector on the gauge theory side (see section 2). It is for instance evident that a mode with J_3 non-zero would get an infinite potential (3.28) of order $1/\lambda$ just as having a gauge theory state with non-zero J_3 also would be of order $1/\lambda$.

It would be interesting to add fermions to the Lagrangian (3.27). This can be done using the approaches of Metsaev and Tseytlin [13, 14] or Frolov et al [15]. We expect that one can show the decoupling of the fermions in a similar way as the above argument for the bosonic directions.

Quantum effects from transverse modes and fermions. As shown above all the modes which do not give a finite contribution have a confining potential which freezes them and confines them to a point in the $\lambda \to 0$ limit. However, they can still contribute through quantum corrections, as we now discuss.

One possible source of quantum corrections is from the zero-modes of the transverse modes and the fermions. We have shown above that near the point where the decoupled modes should be confined to, the modes have a harmonic oscillator potential. Hence, the zero point energy could contribute. However, here we are saved by the fact that we are close to a supersymmetric BPS state. We are considering states with energies slightly above $E = J$, and $E = J$ is a half BPS state. Therefore, the zero-point energy is cancelled out by supersymmetry.

There is another possible source of quantum corrections. Since the transverse modes and the fermions become arbitrarily heavy in the decoupling limit (1.2) we can integrate out

these modes and obtain an effective sigma-model action for the surviving modes. Classically we have shown that we obtain the effective action (3.23) for the surviving modes. However, when taking quantum effects into account, in integrating out the transverse modes and fermions, the action (3.23) can receive corrections. This is possible because the decoupled modes can contribute when we go off-shell. We now consider how such corrections to the action (3.23) should appear.

We first remark that for J large the states which have finite size on the two-sphere parameterized by θ and φ have a large value for the action $I(h, x, S)$ as we can see from (3.23). Therefore, such string states are semi-classical and the quantum corrections are suppressed. This means that in the full effective action obtained by integrating out the decoupled modes the part proportional to J should be given by (3.23). From this we can conclude that we have matched gauge theory and string theory, on the level of the sigma-model model actions, for the part of the sigma-model action which is proportional to J. This is one of the main results of this paper. It means that we can reliably match semi-classical states with large J found from weakly coupled gauge theory in the limit (1.2) to semi-classical states found on the string side in the same limit (1.2).

To go on, we should consider terms in the effective sigma-model action which go like powers of $1/J$, as compared to the leading part (3.23). That $1/J$ is the effective expansion parameter is clear from the fact that $1/J$ is seen to be the effective α' in the leading action (3.23). Moreover, one can show by considering string states on the pp-wave background considered in [27] that the next correction arises as a $1/J$ correction and the higher corrections furthermore come in powers $1/J^n$ [36]. This is done by an analysis similar to the one of Callan et al [17, 18]. It is also clear from this analysis that no off-shell contribution from the decoupled modes can enter at order $1/J$. This is basically because computing the corrections to the surviving string states takes the form of quantum mechanical perturbation theory, with $1/J$ being the perturbation parameter. Since in quantum mechanical perturbation theory it is only at the second order that one can receive contributions from off-diagonal elements of the perturbation we can infer that it is only at order $1/J^2$ that we get off-shell contributions from the decoupled modes. From these considerations we can conclude that the first correction to the action (3.23) is of order $1/J^2$. We can write this as

$$I = \frac{J}{4\pi} \int d\tilde{t} \int_0^{2\pi} d\sigma \left[\sin\theta \, \dot{\varphi} - \frac{1}{4}\Big((\theta')^2 + \cos^2\theta (\varphi')^2 \Big) + \frac{G[\theta, \varphi]}{J^2} + \mathcal{O}(J^{-3}) \right] \qquad (3.29)$$

where $G[\theta, \varphi]$ is a function of θ, φ and their derivatives. We see from this that we can match gauge theory and string theory, on the level of the sigma-model action, for the part of the sigma-model action which is proportional to J plus the part proportional to $J \cdot 1/J = 1$. Instead at order $J \cdot 1/J^2 = 1/J$ the decoupled modes can give new contributions to the effective action, as parametrized by $G[\theta, \varphi]$ in (3.29).

Comparing to the sigma-model action (2.17) derived on the gauge theory side from the ferromagnetic Heisenberg spin chain we can now conjecture how the full effective action for the surviving modes should look. At order J^{1-2n} we get a contribution with $2n$ derivatives with respect to σ such that the full effective action matches the action (2.17) on the gauge

theory side. That integrating out the decoupled modes gives rise to higher-derivative terms is natural. It is interesting to note that each new derivative ∂_σ comes with a $1/J$. This could seem surprising since in (3.23) $1/J$ plays the role of α'. However, in the free limit (2.20) of the sigma-model it is not hard to check that the density of the world-sheet momentum goes like $1/J$ which means that the operator for the world-sheet momentum is proportional to $-(i/J)\partial_\sigma$.

In conclusion we have found that to order $J \cdot 1/J^2 = 1/J$ the sigma-model (3.23) is an accurate description for the surviving modes of type IIB string theory on $\text{AdS}_5 \times S^5$ in the decoupling limit (1.2). This leading part of the sigma-model action (see also (3.29)) matches the leading part of the sigma-model action (2.19) on the gauge theory side, thus giving agreement between string theory and gauge theory in the decoupling limit (1.2) on both sides of the AdS/CFT correspondence, for the leading and first subleading order in an expansion in $1/J$. Furthermore, we conjecture that the decoupled modes can be integrated out on the string theory side to obtain the full sigma-model action (2.17) that is found on the gauge theory side as an effective action for the surviving string modes.

Below in section 3.3 and further in section 4 we examine the consequences of the above results for the matching of the spectra of gauge theory and string theory in the AdS/CFT correspondence.

3.3 Matching of gauge and string theory spectra

In the above we have found that we can match gauge theory and string theory in terms of a sigma-model description in the decoupling limit (1.2) of the AdS/CFT correspondence for J large. We now employ this to match the spectra for gauge theory and string theory in the limit (1.2).

First-quantized string states. On the gauge theory side, we have that the low energy spectrum of $\tilde{H}$ for large J is given by (2.11). We now argue that we can find the same spectrum on the string side in the same regime. Consider the sigma-model action (3.29). As for the sigma-model on the gauge theory side we zoom in near the point $(\theta, \varphi) = (0, 0)$ on the two-sphere by taking the large J limit with θ and φ given by (see eqs. (2.20) and (2.27))

$$x = \sqrt{J}\varphi, \quad p_x = \frac{\sqrt{J}}{4\pi}\sin\theta \tag{3.30}$$

This gives the Hamiltonian (2.28) that we obtained on the gauge theory side, valid up to $1/J^2$ corrections. Note here that $\tilde{H} = i\partial_{\tilde{t}}$ as defined above in section 3.1. Therefore, using the same procedure as in section 2.2 we obtain again the spectrum (2.11). This means that up to order $1/J^2$, we find the same string spectrum from the decoupling limit (1.2) of strings on $\text{AdS}_5 \times S^5$ as we found on the gauge theory side.

Note that the matching of the spectrum here is made in a $\lambda \to 0$ limit with J large but finite. Therefore, in this sense we have that the string sigma-model action $I[h, x, S]$ in (3.25) is finite after the decoupling limit.

We can thus conclude that our matching of the string theory sigma-model action (3.29) to the leading order part of the gauge theory sigma-model action (2.17) enables us to match the first-quantized string spectrum to the spectrum (2.11) found on the gauge theory side.

We emphasize that this matching of spectra is highly non-trivial in that on the string side we are taking a $R^2/\alpha' \to 0$ limit, which ordinarily would mean that the quantum corrections would become large. Instead, in the limit (1.2) we have shown in section 3.2 that we can keep the quantum corrections under control by having J large. A crucial part of this, shown in section 3.1, is that even though the string sigma-model action $I[h, x, S]$ is proportional to R^2/α' there is another part of the action multiplying this that diverges for $R^2/\alpha' \to 0$ such that we end up with a finite action $I[h, x, S]$ in the (1.2) limit.

Another related reason that the matching works is that on the gauge theory side we are able to take a strong coupling limit even though $\lambda \to 0$. This is due to the fact that the effective coupling in the regime (1.1) is not λ but rather $\lambda/(E - J)$. Therefore by having $\lambda/(E - J) \gg 1$ while $\lambda \to 0$ we are accessing a strong coupling regime of the gauge theory side, even though the 't Hooft coupling λ is small. We discuss how to see that $\lambda/(E - J) \gg 1$ is a strong-coupling regime in section 7 and in the conclusion in section 8.

Semi-classical string states. As already anticipated in the end of section 3.1, we can match the leading order contribution to the energy of a semi-classical string state in the decoupling limit (1.2) of type IIB string theory on $\mathrm{AdS}_5 \times S^5$ (provided of course the semi-classical string state survives the limit (1.2)) to the leading order energy of the corresponding state on the gauge theory side. This follows from the matching of the string theory and gauge theory sigma-model actions (3.29) and (2.19) to leading order for $J \to \infty$. That quantum corrections cannot alter this result is due to the fact that $I[h, x, S]$ is of order J for a semi-classical string state.

Again, the matching of the classical energy of a string state to the energy of a gauge theory state is rather non-trivial since we are considering a $R^2/\alpha' \to 0$ limit in (1.2). Thus the lesson here is that we can have a large string sigma-model action $I[h, x, S]$ even though $\lambda \ll 1$. That this is possible follows from the classical limit in section 3.1 where we saw that even though $R^2/\alpha' \to 0$ we still end up with a finite action.

As a simple example of a semi-classical state we can consider the rigid circular string solution [19]

$$\theta = 0\,, \quad \varphi = 2m\sigma \tag{3.31}$$

This corresponds to having $\phi_1 = -\phi_2 = m\sigma$ and $\psi = \pi/4$. The classical energy of this state is $\tilde{H} = Jm^2/2$.

Note that our results for the matching of the actions on the string side (3.29) and the gauge theory side (2.19) also means that one can match $1/J$ corrections to the leading classical result for the energy. This provides an explanation to the many spectacular results for the matching of energies for semi-classical states [19 – 21].

4. Connection to semi-classical string regime

In section 3 we have shown that we can match the leading spectra of gauge and string theory by taking the limit (1.2) on both sides of the AdS/CFT correspondence. In this section we argue that our results explains the matching between weakly-coupled gauge theory and string theory in the semi-classical regime at first order in $\lambda' = \lambda/J^2$.

The argument is rather simple. In section 2 and 3 we have examined gauge theory and string theory in the regime

$$\lambda \ll 1, \quad J \gg 1 \tag{4.1}$$

In this regime we can expand the energy/scaling dimension in λ and $1/J$ on both the gauge theory and the string theory side. Expanding in λ we can write the gauge theory energy/scaling dimension as

$$E_{\rm gt} - J = \lambda A_1 + \lambda^2 A_2 + \mathcal{O}(\lambda^3) \tag{4.2}$$

and the energy of string states as

$$E_{\rm str} - J = \lambda B_1 + \lambda^2 B_2 + \mathcal{O}(\lambda^3) \tag{4.3}$$

At each order in λ we can then expand in $1/J$. At first order in λ we have

$$A_1 = \frac{a_1}{J^2} + \frac{a_2}{J^3} + \mathcal{O}(J^{-4}), \quad B_1 = \frac{b_1}{J^2} + \frac{b_2}{J^3} + \mathcal{O}(J^{-4}) \tag{4.4}$$

Validity of the AdS/CFT correspondence in (4.1) means that $E_{\rm gt} = E_{\rm str}$. Indeed, we have shown in section 3 that $a_1 = b_1$ and $a_2 = b_2$ in the SU(2) sector. This result is thus a non-trivial confirmation of the AdS/CFT correspondence and it relied on the result (3.29) which shows that quantum corrections to the classical sigma-model only enters at order $1/J^2$ as compared to the leading term.

Consider instead the regime

$$1 \ll \lambda \ll J^2 \tag{4.5}$$

This is a semi-classical regime for type IIB string theory since $\lambda \gg 1$. In this regime we can expand the energy of string states in $\lambda' = \lambda/J^2$ [11]

$$E_{\rm str} - J = \lambda' C_1 + \lambda'^2 C_2 + \mathcal{O}(\lambda'^{5/2}) \tag{4.6}$$

At each order in λ' we can then expand in powers of $1/J$

$$C_1 = c_1 + \frac{c_2}{J} + \mathcal{O}(J^{-2}) \tag{4.7}$$

Since the two regimes (4.1) and (4.5) do not overlap there is a priori no reason why the energies should agree in these two regimes. Indeed a mismatch can be resolved by introducing an interpolating function of λ between the two.[12] However, it has been found in numerous computations, both for quantum string states and semi-classical string states, that $a_1 = c_1$ and $a_2 = c_2$ [9, 16, 40, 17–21]. This agreement begs for an explanation.

We can argue for $a_1 = c_1$ and $a_2 = c_2$ as follows. In the regime (4.1) we can infer from (3.29) that the computation of b_1 and b_2 only involves the classical Landau-Lifshitz

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[11] Note that in this semi-classical regime there are also non-analytical terms in λ contributing [37, 38]. These terms comes from quantum corrections to the classical string result and can be seen as $1/\sqrt{\lambda}$ corrections to (4.6) and are therefore small when λ is large.

[12] In the study of integrability of the AdS/CFT correspondence this has been achieved by the introduction of a phase-factor which changes as one goes from $\lambda \ll 1$ to $\lambda \gg 1$ [25, 39, 26].

sigma-model. In the semi-classical regime (4.5) we have the classical sigma-model limit for $\lambda \gg 1$. Therefore, to leading order we can compute the energy of string states from the classical Landau-Lifshitz sigma-model. This is in particular true for the c_1 and c_2 coefficients which are not affected by quantum corrections. Therefore, it follows that $b_1 = c_1$ and $b_2 = c_2$ since these coefficients are computed from the classical Landau-Lifshitz sigma-model in both regimes. This agreement holds both for semi-classical string states, with a large number of excitations, as well as for quantum string states. Since $a_1 = b_1$ and $a_2 = b_2$ as consequence of our results in section 3, we see that *it follows from our results that $a_1 = c_1$ and $a_2 = c_2$ in the* $\mathrm{SU}(2)$ *sector*. Using our argument of section 3 we have thus bridged the gap between the two non-overlapping regimes (4.1) and (4.5) by connecting the two regimes on the string theory side. We can therefore conclude that the agreement of weakly coupled gauge theory and semi-classical string theory at one-loop up to $1/J^2$ corrections (*i.e.* $a_1 = c_1$ and $a_2 = c_2$) is not a coincidence.

It has furthermore been found that $J^4 A_2 = C_2$ for the leading and first order correction in $1/J$ [16–18]. It would be very interesting if one could extend our arguments to get an understanding of this agreement as well.

5. The decoupling limit as a non-relativistic limit

In this section we show that the decoupling limit (1.2) corresponds to a non-relativistic limit of type IIB string theory on $\mathrm{AdS}_5 \times S^5$. We show this in full detail for type IIB string theory on the maximally supersymmetric pp-wave in section 5.1. In section 5.2 we extend the analysis to the $\mathrm{AdS}_5 \times S^5$ background and we comment on the relation to other non-relativistic limits in string and M-theory.

5.1 Non-relativistic limit of string theory on pp-wave

Penrose limit for flat-direction pp-wave. We review here how to take the Penrose limit of [27], giving the maximally supersymmetric pp-wave background of [28] in a coordinate system with a flat direction [29, 27]. The $\mathrm{AdS}_5 \times S^5$ background is (3.1) and (3.2). For simplicity, we ignore the five-form field strength and the fermionic fields in the following. Using the variables (3.6) along with

$$t' = t \,, \quad \chi = \phi_+ - t \tag{5.1}$$

and using (3.26), we get the following metric for $\mathrm{AdS}_5 \times S^5$

$$ds^2 = R^2 \cos^2 \zeta \left[2dt' d\chi + \sin \theta d\varphi dt' + \frac{1}{4}(d\theta^2 + \cos^2 \theta d\varphi^2) + \left(d\chi + \frac{1}{2} \sin \theta d\varphi \right)^2 \right]$$
$$- R^2 (\sinh^2 \rho + \sin^2 \zeta)(dt')^2 + R^2 \left[d\rho^2 + \sinh^2 \rho (d\Omega_3')^2 + d\zeta^2 + \sin^2 \zeta d\alpha^2 \right] \tag{5.2}$$

Here $E - J = i\partial_{t'}$, $J = -i\partial_\chi$ and $S_z = -i\partial_\varphi$. Define now the coordinates γ, x, y, r and $\tilde{r}$ by

$$\gamma = J\chi \,, \quad x = \sqrt{J}\varphi \,, \quad y = \sqrt{J}\theta \,, \quad r = \sqrt{J}\rho \,, \quad \tilde{r} = \sqrt{J}\zeta \tag{5.3}$$

Then the Penrose limit is

$$J \to \infty\,, \quad \lambda' \equiv \frac{\lambda}{J^2} \text{ fixed}\,, \quad \alpha' \text{ fixed}\,, \quad t',\, \gamma,\, x,\, y,\, r,\, \tilde{r},\, \alpha,\, \Omega_3' \text{ fixed} \tag{5.4}$$

Taking the Penrose limit gives the metric

$$\frac{ds^2}{\alpha'\sqrt{\lambda'}} = 2dt'd\gamma + \frac{1}{4}(dx^2 + dy^2) + ydxdt' + \sum_{i=1}^{6} dz_i^2 - \sum_{i=1}^{6} z_i^2 (dt')^2 \tag{5.5}$$

Here the coordinates $z_1, \ldots, z_4$ are defined by $r^2 = \sum_{i=1}^{4} z_i^2$ and $dr^2 + r^2(d\Omega_3')^2 = \sum_{i=1}^{4} dz_i^2$ and z_5, z_6 are defined by $z_5 + iz_6 = \tilde{r}e^{i\alpha}$. We see that this is the pp-wave background considered in [29, 27]. Choosing the gauge

$$t' = c\tau\,, \quad h_{\alpha\beta} = \eta_{\alpha\beta} \tag{5.6}$$

we obtain the gauge fixed Lagrangian

$$\mathcal{L}_{\text{gf}} = \frac{c}{2}y\partial_\tau x + \frac{(\partial_\tau x)^2 + (\partial_\tau y)^2}{8} - \frac{x'^2 + y'^2}{8} + \frac{1}{2}\sum_{i=1}^{6}\left[(\partial_\tau z_i)^2 - z_i'^2 - c^2 z_i^2\right] \tag{5.7}$$

along with the action

$$I = \frac{\sqrt{\lambda'}}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \mathcal{L}_{\text{gf}} \tag{5.8}$$

From the term $c\partial_\tau \gamma$ in the full Lagrangian, the constant c can be fixed to be

$$c = \frac{1}{\sqrt{\lambda'}} \tag{5.9}$$

The Hamiltonian is

$$H_{\text{lc}} = \frac{\lambda'}{2\pi}\int_0^{2\pi} d\sigma \left\{ \frac{(\partial_\tau x)^2 + (\partial_\tau y)^2}{8} + \frac{x'^2 + y'^2}{8} + \frac{1}{2}\sum_{i=1}^{6}\left[(\partial_\tau z_i)^2 + z_i'^2 + c^2 z_i^2\right] \right\} \tag{5.10}$$

Defining $z(\tau,\sigma) = x(\tau,\sigma) + iy(\tau,\sigma)$, we can write the mode expansions of the bosonic fields as

$$z(\tau,\sigma) = 2\sqrt{c}\, e^{ic\tau} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{\omega_n}}\left[a_n e^{-i(\omega_n\tau - n\sigma)} - \tilde{a}_n^\dagger e^{i(\omega_n\tau - n\sigma)} \right] \tag{5.11}$$

$$z_i(\tau,\sigma) = i\frac{\sqrt{c}}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{\omega_n}}\left[a_n^i e^{-i(\omega_n\tau - n\sigma)} - (a_n^i)^\dagger e^{i(\omega_n\tau - n\sigma)} \right] \tag{5.12}$$

where we used $\omega_n = \sqrt{n^2 + c^2}$. The canonical commutation relations $[x(\tau,\sigma), p_x(\tau,\sigma')] = i\delta(\sigma - \sigma')$, $[y(\tau,\sigma), p_y(\tau,\sigma')] = i\delta(\sigma - \sigma')$ and $[z_i(\tau,\sigma), p_j(\tau,\sigma')] = i\delta_{ij}\delta(\sigma - \sigma')$ follows from

$$[a_m, a_n^\dagger] = \delta_{mn}\,, \quad [\tilde{a}_m, \tilde{a}_n^\dagger] = \delta_{mn}\,, \quad [a_m^i, (a_n^j)^\dagger] = \delta_{mn}\delta_{ij} \tag{5.13}$$

Employing (5.13) we obtain the bosonic spectrum

$$cH_{\text{lc}} = \sum_{n\neq 0}(\omega_n - c)M_n + \sum_{n\in\mathbb{Z}}(\omega_n + c)N_n + \sum_{i=1}^{6}\sum_{n\in\mathbb{Z}}\omega_n N_n^i \tag{5.14}$$

with the number operators $M_n = a_n^\dagger a_n$, $N_n = \tilde{a}_n^\dagger \tilde{a}_n$ and $N_n^i = (a_n^i)^\dagger a_n^i$, and with the level-matching condition

$$\sum_{n \neq 0} n M_n + \sum_{n \in \mathbb{Z}} n N_n + \sum_{i=1}^{6} \sum_{n \in \mathbb{Z}} n N_n^i = 0 \qquad (5.15)$$

Decoupling limit as non-relativistic limit. We now consider the decoupling limit (1.2) and show explicitly that it is a non-relativistic limit.

As in [6] we can take the limit directly of the spectrum (5.14). We first notice that the rescaled energy defined in (2.10) is $\tilde{H} = c^2 H_{\text{lc}}$. Since $c \to \infty$ we see that the modes with non-zero N_n and N_n^i become infinitely heavy, whereas the modes with M_n gives the following spectrum

$$\tilde{H} = \frac{1}{2} \sum_{n \neq 0} n^2 M_n \,, \quad \sum_{n \neq 0} n M_n = 0 \qquad (5.16)$$

This match the leading part of (2.11), in accordance with sections 3 and 4 [6].

We see that the decoupling of the modes N_n^i corresponds to the decoupling of the six modes transverse to $\mathbb{R} \times S^3$ as discussed in section 3.2. Instead for the N_n modes, we can interpret their decoupling as a consequence of the non-relativistic nature of the limit (1.2), as we now shall discuss.

The first hint that the limit (1.2) is non-relativistic comes from considering the dispersion relation for the M_n modes. Notice that a single mode with $M_n = 1$ has

$$E - J = \sqrt{1 + \lambda' n^2} - 1 \qquad (5.17)$$

We can interpret this as a relativistic dispersion relation $\mathcal{E} = \sqrt{m^2 + p^2}$ where the energy is $\mathcal{E} = (E - J + 1)/\sqrt{\lambda'}$, the rest-mass is $m = 1/\lambda'$ and the momentum is $p = n$. The non-relativistic limit is then that $p/m \to 0$ giving a Galilean dispersion relation $\mathcal{E} - m = p^2/(2m)$. We see that this precisely is realized by the limit (1.2).

Consider now the mode expansion (5.11) of the field $z(\tau, \sigma) = x(\tau, \sigma) + iy(\tau, \sigma)$. Before taking the limit we see that we have two sets of modes $a_{n \in \mathbb{Z}}$ and $\tilde{a}_{n \in \mathbb{Z}}$. This is in accordance with having two spatial directions. Considering the limit (1.2) of (5.11), we see that

$$z(\tau, \sigma) = 2 \sum_{n \in \mathbb{Z}} \sqrt{\frac{c}{\omega_n}} \Big[a_n e^{-i(\omega_n - c)\tau + in\sigma} - \tilde{a}_n^\dagger e^{i(\omega_n + c)\tau - in\sigma} \Big]$$

$$\simeq 2 \sum_{n \in \mathbb{Z}} a_n e^{-i\frac{n^2}{2}\tilde{t} + in\sigma} - 2 \sum_{n \in \mathbb{Z}} \tilde{a}_n^\dagger e^{2ic^2\tilde{t} - in\sigma} \qquad (5.18)$$

where we used the rescaled time $\tilde{t} = \lambda' t' = \sqrt{\lambda'}\tau$, as introduced in section 3.1. The time $\tilde{t}$ measures the appropriate energy scale $\tilde{H}$ for the limit (1.2). The first term in (5.18) clearly becomes the mode expansion for the surviving modes after the limit, corresponding to the mode expansion (2.22) for the free limit of the sigma-model describing the Heisenberg spin chain, see section 2.2. The second term in (5.18) decouples since $c^2 \to \infty$. Therefore, after the limit only the $a_{n \in \mathbb{Z}}$ modes are left. That half of the modes vanishes when going from a relativistic to a Galilean dispersion relation has a clear physical interpretation. This being

that before the limit any particle mode has an anti-particle mode propagating backwards in time, as is the case in a relativistic field theory. After the limit we instead have Galilean symmetry, now with the anti-particle modes $\tilde{a}_{n\in\mathbb{Z}}$ decoupled. *I.e.* the field $z(\tilde{t}, \sigma)$ only contains lowering operators after the limit and has thus no anti-particle part.

We can furthermore see that the limit (1.2) is non-relativistic by considering the velocities. We have that $\tau = c\tilde{t}$ and hence the velocities $\partial_\tau x$, $\partial_\tau y$ and ∂z_i all go to zero like $1/c$ as $c \to \infty$. That the velocities go to zero is obviously a clear signature of a non-relativistic limit. Taking the $c \to \infty$ limit of the Lagrangian (5.7) we get

$$\mathcal{L}_{\text{gf}} = \frac{1}{2}y\partial_{\tilde{t}}x - \frac{x'^2 + y'^2}{8} - \frac{1}{2}\sum_{i=1}^{6}\left[z_i'^2 + c^2 z_i^2\right] \tag{5.19}$$

We see that only $x(\tilde{t}, \sigma)$ is dynamical after the limit. The six transverse directions $z_i(\tau, \sigma)$ are non-dynamical and decoupled, and the potential forces the string to be located at $z_i = 0$ which is also what we found in section 3.2. Taking the limit (1.2) on the level of the action, we thus get the action

$$I = \frac{1}{2\pi}\int d\tilde{t}\int_0^{4\pi} d\sigma\left[y\partial_{\tilde{t}}x - \frac{x'^2 + y'^2}{4}\right] \tag{5.20}$$

This precisely correspond to the action (2.21) obtained in the free limit of the sigma-model found on the gauge theory side. In section 6 we consider further the two ways of obtaining the action (5.20). As discussed in section 2.2 the action (5.20) is a theory with Galilean symmetry and spectrum (5.16).

Consider the momenta

$$p_x = \frac{1}{4\pi}y + \frac{1}{8\pi c}\partial_\tau x\,, \quad p_y = \frac{1}{8\pi c}\partial_\tau y \tag{5.21}$$

Since $\partial_\tau x$ and $\partial_\tau y$ both go to zero as $1/c$ when $c \to \infty$, we have that $p_x \to y/(4\pi)$ and $p_y \to 0$ in the limit (1.2). From the canonical commutator $[x(\tau, \sigma), p_x(\tau, \sigma')] = i\delta(\sigma - \sigma')$ we see now that

$$[x(\tau, \sigma), y(\tau, \sigma')] = 4\pi i\delta(\sigma - \sigma') \tag{5.22}$$

Thus, x and y become non-commutative in the limit (1.2). Note here that before the limit we have that $[x(\tau, \sigma), y(\tau, \sigma')] = 0$. However, the origin of this non-commutativity is the decoupling of the $\tilde{a}_{n\in\mathbb{Z}}$ modes.

The non-commutativity (5.22) connects also to another aspect of the non-relativistic nature of the limit (1.2). As one can see for example from (5.16) we have after the limit a one-dimensional Galilean theory, instead of the two-dimensional relativistic theory before the limit. Thus, we effectively go from having two spatial directions x and y, to having only one spatial direction x in the limit (1.2). Since $p_x \to y/(4\pi)$ in the limit we see that the two spatial directions become the directions in a two-dimensional phase space for a single spatial direction. In this way we accomplish reducing the dimension from two to one. This ties up with the non-commutativity of x and y in (5.22) since now y is the momentum conjugate of x.

We discuss the relation to the literature on non-relativistic limits and string theories with Galilean symmetries below in section 5.2.

5.2 General considerations

In section 5.1 we have shown explicitly that the limit (1.2) of type IIB string theory on the maximally supersymmetric pp-wave background (5.5) is a non-relativistic limit. We now explain that the limit (1.2) of type IIB string theory on $\text{AdS}_5 \times S^5$ also corresponds to a non-relativistic limit. To see this consider the limit in section 3.1. From (3.12) and (3.18) we see that for the fields $\varphi(\tau, \sigma)$ and $\theta(\tau, \sigma)$ the velocities $\partial_\tau \varphi$ and $\partial_\tau \theta$ go to zero since $\kappa \to 0$ in the limit (1.2). This is obviously a clear sign of taking a non-relativistic limit and is in close resemblance with the limit of the pp-wave considered in section 5.1. If we consider the momenta conjugate to φ and θ we have

$$p_\varphi = \frac{J}{4\pi} \sin\theta + \frac{\sqrt{\lambda}}{8\pi} \partial_\tau \varphi + \frac{\sqrt{\lambda}}{4\pi} \partial_\tau \chi \,, \quad p_\theta = \frac{\sqrt{\lambda}}{8\pi} \partial_\tau \theta \tag{5.23}$$

We get from this that $p_\varphi \to J \sin\theta/(4\pi)$ and $p_\theta \to 0$ in the limit (1.2). Using the same reasoning as in section 5.1 we see that we go from a theory with two spatial directions φ and θ to a one-dimensional theory in which φ and θ instead parameterizes the phase space. This is in accordance with the Landau-Lifshitz action (3.23). Moreover, the above limit of the momenta p_φ and p_θ shows that φ and θ do not commute after the limit, in accordance with the analysis of section 2.2. In conclusion, the limit (1.2) is a non-relativistic limit of type IIB string theory on $\text{AdS}_5 \times S^5$.

It is interesting to compare our non-relativistic limit of type IIB string theory on $\text{AdS}_5 \times S^5$ to other non-relativistic limits of string and M-theory which have been considered previously. For a D-brane with a near-critical electric field it has been found that the open strings on the D-brane result in an open string theory with Galilean dynamics and space-time non-commutativity, while the closed string sector decouples [41–45] and this has furthermore been generalized to other branes as well [46–48]. Building on these limits, it was found in [49–51] that one can find non-relativistic closed string theories (NRCS's) with Galilean dynamics by taking a near-critical limit of string theory in the background of a near-critical field in which one of the directions compactified. A similar NRCS limit was furthermore found for type IIB string theory on $\text{AdS}_5 \times S^5$ [52].

The features common between the NRCS limits and our limit (1.2) are that they are low energy limits, which in our case corresponds to sending $E - J \to 0$, they are non-relativistic limits, i.e. limits of slow velocities and with Galilean dynamics, and they have modes that become infinitely heavy in the limit and therefore decouple. It seems on the other hand that there is not any direct relation or duality between our limit (1.2) and the NRCS limits since we find a space-space non-commutative target space and since in the NRCS theories a compact direction is needed in order to take the limits (the surviving closed strings all have non-zero winding). Moreover, we find a truncation of the degrees of freedom of the theory in that the dimension of the target space is reduced, in contrast with the NRCS limits in which the dimension of the target space is preserved.

6. Decoupling limit versus Penrose limit

We consider in this section briefly the interplay between two different kinds of limits that one can take of type IIB string theory on $\text{AdS}_5 \times S^5$. The first kind is the Penrose limit.
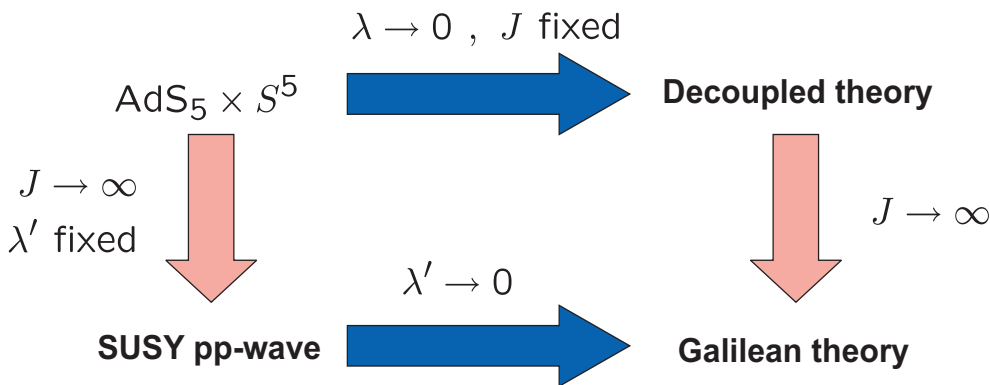
Figure 1: Overview of the limits. The limits going downwards corresponds to the Penrose limit and its manifestation for the decoupled theory. The limits going from left to right corresponds to the decoupling limit (1.2) and its manifestation for the maximally supersymmetric pp-wave background.

This is purely geometrical limit in which the number of degrees of freedom is the same as before the limit, but the background of the strings changes from $AdS_5 \times S^5$ to the maximally supersymmetric pp-wave background of [28]. The second kind of limit is the decoupling limit (1.2), see also [7] for other limits of this kind. This kind of limit is not geometrical but is taken directly of the string theory, as we did in section 3 for the $AdS_5 \times S^5$ sigma-model. In this kind of limit we zoom in on a particular regime of the theory in which some of the degrees of freedom decouple from the spectrum due to the rescaling of the energy and moreover the interactions between the surviving modes simplify.

In figure 1 we have illustrated the four limits that we are considering. The top limit going from left to right is the decoupling limit (1.2) that we considered in section 3. The left limit going downwards is the Penrose limit of $AdS_5 \times S^5$ [27] giving the maximally supersymmetric pp-wave background of [28]. We review briefly this limit in section 5.1. Note that the relevant Penrose limit is the one of [27] rather than the one of [53, 54]. We considered in section 5.1 the limit on figure 1 on the bottom going from left to right. This limit is the manifestation of the decoupling limit for the maximally supersymmetric pp-wave background and was found in [6] (see also [8]). Finally, the right limit going downwards is the manifestation of the pp-wave limit for the decoupled theory. This limit is discussed in section 2.2 where it is shown that we end up with a one-dimensional non-relativistic theory with a Galilean symmetry, hence the name "Galilean theory".

As depicted in figure 1 one obtains the same "Galilean theory" irrespective of whether one first takes the Penrose limit and then the decoupling limit or vice versa. Indeed this follows from the arguments of sections 3 and 4. That the two limits in this sense commute, in that you end up with the same end point, is non-trivial since if we first take the pp-wave limit then $\lambda \to \infty$ as part of that limit, whereas if we take the other way around the diagram in figure 1 then we always keep λ small.

7. Hagedorn temperature as interpolating quantity

The limit (1.2) was originally conceived as a limit of the Grand Canonical partition function of $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. For our purposes we can write this partition function as $Z(T, \Omega)$, where T is the temperature and Ω is the chemical potential associated to $J = J_1 + J_2$. The limit in the Grand Canonical ensemble takes the form [5, 6]

$$\Omega \to 1 \,, \quad \tilde{T} \equiv \frac{T}{1 - \Omega} \text{ fixed} \,, \quad \tilde{\lambda} \equiv \frac{\lambda}{1 - \Omega} \text{ fixed} \,, \quad N \text{ fixed} \tag{7.1}$$

The resulting partition function is [5, 6]

$$Z(\tilde{\beta}) = \text{Tr}\left[e^{-\tilde{\beta}(D_0 + \tilde{\lambda}D_2)} \right] \tag{7.2}$$

where the trace is only over the SU(2) sector and $\tilde{\beta} = 1/\tilde{T}$. In the planar limit $N = \infty$ one finds [5, 6]

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{J=1}^{\infty} \frac{1}{n} e^{-\tilde{\beta}nL} Z_J^{(\text{XXX})}(n\tilde{\beta}) \tag{7.3}$$

where $Z_J^{(\text{XXX})}(\tilde{\beta})$ is the partition function for the ferromagnetic Heisenberg spin chain of length J with Hamiltonian $\tilde{\lambda}D_2$. Thus, as in the microcanical ensemble, planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the limit (7.1) is given exactly by the Heisenberg spin chain. Using this it was found in [6] that the Hagedorn temperature is determined from the thermodynamic limit of the free-energy per site of the ferromagnetic Heisenberg spin chain

$$f(t) = -t \lim_{J \to \infty} \frac{1}{J} \log\left[\text{Tr}_J\left(e^{-t^{-1}D_2} \right) \right] \tag{7.4}$$

from the formula

$$f\left((\tilde{\beta}_H \tilde{\lambda})^{-1} \right) = -\tilde{\lambda}^{-1} \tag{7.5}$$

Using (7.5) one can find the Hagedorn temperature $\tilde{T}_H(\tilde{\lambda})$ as function of $\tilde{\lambda}$. This was done in [6] both for $\tilde{\lambda} \ll 1$ and $\tilde{\lambda} \gg 1$. More generally, we can infer from (7.5) that we can interpolate the Hagedorn temperature $\tilde{T}_H$ from $\tilde{\lambda} \ll 1$ to $\tilde{\lambda} \gg 1$.

For $\tilde{\lambda} \ll 1$ we can connect to the loop corrections in weakly coupled $\mathcal{N} = 4$ SYM. More specifically, for $\tilde{\lambda} \ll 1$ each term of power $\tilde{\lambda}^k$ in $\tilde{T}_H$ origins from a k-loop correction in weakly coupled $\mathcal{N} = 4$ SYM [6]. Therefore, if we instead consider $\tilde{\lambda} \gg 1$ we can infer that this is a strong-coupling regime of $\mathcal{N} = 4$ SYM, even though we have $\lambda \ll 1$. Having $\tilde{\lambda} \gg 1$ is equivalent to having $\lambda/(E - J) \gg 1$ in the microcanonical ensemble.[13] Thus we can conclude that $\lambda/(E - J) \gg 1$ corresponds to a strong coupling regime of $\mathcal{N} = 4$ SYM, even though $\lambda \ll 1$.

For $\tilde{\lambda} \gg 1$ it was found that $\tilde{T}_H = (2\pi)^{1/3} \zeta(3/2)^{-2/3} \tilde{\lambda}^{1/3}$ [6]. This result is obtained from the spectrum (2.26). Since we have shown in this paper that this spectrum can be

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[13] To see this in detail, one uses that $1 - \Omega$ sets the energy scale in the limit (7.1), *i.e.* $\beta(E - J) = \tilde{\beta}(E - J)/(1 - \Omega)$, so states which contribute to the partition function has $E - J \lesssim 1 - \Omega$. Hence $\tilde{\lambda} = \lambda/(1 - \Omega) \gg 1$ is translated in the microcanonical ensemble to $\lambda/(E - J) \gg 1$.

found both from the gauge theory side as well as the string theory side of AdS/CFT in the limit (1.2) we can match this Hagedorn temperature to the Hagedorn temperature of type IIB string theory on $\text{AdS}_5 \times S^5$. This was done in [6]. However, here we justify the steps of [6] in which the Hagedorn temperature was found on the string theory side by first taking the Penrose limit (5.4) and subsequently taking the limit (1.2). Indeed, we have shown in sections 3–6 (see in particular the commuting diagram in figure 1, section 6) that one obtains the same spectrum (2.26) by first taking the limit (1.2) and subsequently considering $J \to \infty$. Therefore, we can conclude that the Hagedorn temperature constitutes the first example of a quantity, not protected by supersymmetry, which we can interpolate fully from weak to strong coupling in AdS/CFT.

8. Conclusions

The basic idea of this paper is that we can compare gauge theory and string theory quantitatively in the regime (1.1) of the AdS/CFT correspondence. The special thing about the regime (1.1) is that the 't Hooft coupling is small which means we can compute the spectrum of states exactly using weakly coupled $\mathcal{N} = 4$ SYM. Ordinarily, this would mean that we are deep in a quantum string regime on the string theory side, since the length scale of the $\text{AdS}_5 \times S^5$ geometry is much smaller than the string length, but we show in this paper that we can find a semi-classical string theory regime as part of the regime (1.1). This is related to the fact that while $\lambda \ll 1$ we have that $\lambda/(E-J) \gg 1$ which effectively means that we are in a strong-coupling regime of $\mathcal{N} = 4$ SYM.

That $\lambda/(E-J) \gg 1$ is a strong-coupling regime of $\mathcal{N} = 4$ SYM is tied to the fact that we can see strings with continuous world-sheets in the regime (1.1). In particular, we find in this paper that there are semi-classical strings with $\lambda/(E-J) \sim J$ and that single quantum strings which are weakly interacting on the world-sheet appear for $\lambda/(E-J) \sim J^2$. As we explain in this paper, it is the ability to match the energies for such strings that makes a quantitative match of the spectra of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ and type IIB string theory on $\text{AdS}_5 \times S^5$ in the regime (1.1) possible.

We explore the regime (1.1) by taking the decoupling limit (1.2). We show that this limit can be seen as a non-relativistic of strings on $\text{AdS}_5 \times S^5$, and it is conjectured to give a consistent truncation of both the gauge theory and string theory sides of the AdS/CFT correspondence. For planar $\mathcal{N} = 4$ SYM and type IIB string theory with $g_s = 0$ we show that one can match the leading terms in a sigma-model action in the decoupling limit (1.2) and for large J. This relies on our identification of a new semi-classical regime for the string side. Employing this, we match the spectra up to $1/J^2$ corrections. We explain that this result shows why it has been found in several ways that the one-loop contribution to the spectrum matches the string theory spectrum up to $1/J^2$ corrections.

The results of this paper give a better understanding of the matching of Hagedorn temperature in [6]. We conclude that the Hagedorn temperature constitutes the first example of a quantity, not protected by supersymmetry, which we can interpolate fully from weak to strong coupling in AdS/CFT.

Given that we have managed to identify a regime of the AdS/CFT correspondence in which we can quantitatively match gauge theory and string theory, it is interesting to ask what future applications this can have. A very interesting direction taken in [36][14] is to match the spectrum of type IIB strings on $AdS_5 \times S^5$ to the spectrum obtained on the gauge theory side at order $1/J^2$ in the limit (1.2). The matching at order $1/J^2$ would be highly interesting in that it should involve a non-trivial contribution coming from integrating out the modes that decouple in the limit (1.2), giving rise to a higher-derivative term in the effective sigma-model description of the strings.

Another direction that one could pursue is to compute λ corrections on the string theory side. This seems challenging on the string theory side since one in principle should integrate out the heavy decoupled modes order by order in λ and then do quantum mechanical perturbation theory in λ. However, if one succeeds it could provide an alternative and more direct path to resolving the famous three-loop discrepancy [17, 18, 55].

Another very interesting avenue to explore is to move away from the planar limit and $g_s = 0$. One direction could be to take a new look at $1/N$ corrections.[15] It is conceivable that the fact that string theory simplifies in the limit (1.2) could help in going further in this direction. Another interesting direction would be to explore the regime of finite string coupling where one should see black holes.[16] Obviously, if one could use the regime (1.1) and the decoupling limit (1.2) to quantitatively match gauge theory with black holes on the string theory side, it would be a result of tremendous importance.

Finally, it is interesting to generalize our results to gauge/string dualities with less supersymmetry. We note in particular the papers [68 – 71] in which the thermodynamics are considered for gauge/gravity correspondences with less than maximal supersymmetry. We remark furthermore that we found a decoupling limit similar to (1.2) for pure Yang-Mills theory in [7].

Acknowledgments

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[14]See also [21].

[15]See [56 – 60] for recent developments in computing $1/N$ corrections to string states.

[16]See [61 – 67] for recent work on connecting the thermodynamics of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with the thermodynamics of black holes in $AdS_5 \times S^5$. Note in particular [66] in which they consider a region with near-critical chemical potential.

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